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# Optimal Regulation and Passivity Results for Axisymmetric Rigid Bodies Using Two Controls

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**We present a partial solution to the problem of optimal feedback reorientation of the symmetry axis of an axially symmetric rigid body. The performance index is quadratic in the state and the control variable, and the optimal reorientation maneuver requires the use of only two control torques. Because of the passivity characteristics and the cascade structure of the system we first state two optimal regulation problems for the dynamics and the kinematics subsystems, separately. In this case one is able to find explicit solutions to the associated Hamilton–Jacobi equations. For the complete system, we present solutions for two partial cases. The first case is when there is no penalty on the control input. In this case, one can asymptotically recover the cost for the kinematics by making the dynamics sufficiently fast. The second case investigates restrictions imposed by optimality considerations on the aforementioned control law to avoid high gain.**

## Introduction

THE optimal control problem of a rigid body has a long history stemming mainly from the interest of aerospace engineers in the control of rigid spacecraft. Several performance indices have been used in the formulation of the optimal control problem.[1–5] The optimal regulation problem has been mainly addressed for the angular velocity equations only, i.e., without any reference to the kinematics in Refs. 6, 7, and, more recently, Ref. 8. Open-loop solutions can be generated using Pontryagin's maximum principle. This results to a two-point boundary-value problem, which is solved using numerical techniques.[2,9–11] Linear quadratic regulator type formulations for the linearized system have also been reported in the literature both for the rigid and the flexible cases.[12] For the nonlinear problem, Carrington and Junkins[13] have used a polynomial expansion approach to approximate the solution to the Hamilton–Jacobi–Bellman equation. Similar results were reported by Dwyer[14] and Dwyer and Sena.[15] Finally, the book by Junkins and Turner[16] provides a comprehensive compilation of most of the existing results on the rigid-body optimal control problem.

In this paper we seek solutions to the optimal feedback regulation problem of an axially symmetric rigid body where both the angular velocity and the orientation of the body are regulated. The purpose of the stabilizing optimal control is to drive the system to its final rest position defined here to be along a specified direction of the symmetry axis. We assume that the relative orientation of the body about the symmetry axis is irrelevant; only the location of the symmetry axis is of interest. This could be the case when the symmetry axis coincides with the boresight or line of sight of a camera or a gun barrel, for example. Clearly, the relative rotation of the camera or the barrel has no influence on the clarity of the photograph or the accuracy of the projectile. Most importantly, spin-stabilized spacecraft also fall into this category.

The work of Dwyer,[14] Dwyer and Sena,[15] and Dwyer[17] has perhaps the closest connection to the results of this paper. They also seek closed-form solutions to the feedback optimal control problem via the Hamilton–Jacobi equation (HJE) method. The main difference with our approach is that Dwyer and Sena apply a linearizing feedback transformation to the equations, resulting in a linear system in double integrator form. The quadratic regulator problem can then be easily solved over either a finite or an infinite time horizon. In the present work we address the nonlinear problem directly. No linearizing transformation is necessary. We rely on the special structure and the passivity properties of the equations to find closed-form solutions to the Hamilton–Jacobi–Bellman equation associated with the optimization problem.

For the axisymmetric case it turns out that the objective of optimal regulation of the symmetry axis can be achieved using only two torques about axes that span the plane perpendicular to the symmetry axis. Therefore, without loss of generality, we restrict ourselves to the two control input case. This configuration does not allow any freedom to change the angular velocity along the symmetry axis. The angular velocity along this axis is fixed to its initial value. An additional third control about the symmetry axis could be used if regulation of the axial component of the angular velocity and/or the orientation about the symmetry axis is desired. Finally, we note in passing that the case of optimal regulation of a general (nonsymmetric) rigid body using three control torques has been addressed elsewhere.[18]

Taking into consideration the cascade interconnection of the system equations, we first state and solve the optimal regulation problem for the kinematics of the attitude motion when the angular velocity acts as a control input. The cost includes a penalty on the orientation parameters and the angular velocity. The fact that the derivation of optimal feedback solutions is possible for the attitude problem has been noticed in the past, and it is related to the Lie group structure of the configuration space of the motion[19] as well as the passivity properties of the system. Actually, we state an intermediate result of independent interest that relates passivity and optimality for general passive (lossless) nonlinear systems.

For the rigid-body problem the actual control input is, of course, the acting torque that enters through the Euler equations (the dynamics). Optimal regulation with the dynamics included in the problem, and for general performance indices, is not yet solved—as far as the author knows. However, the optimization problem for the kinematics provides a lower bound on the achievable performance for the whole system for the same cost functional. Actually, we show that if the dynamics are fast (or can be made fast enough through the appropriate choice of the control input), one is able to recover this performance asymptotically. We show how such a controller can be constructed—and thus achieve the optimal performance—under the assumption that there is no penalty on the control effort. This controller will include, in general, a high-gain portion. Motivated by the optimal characteristics of this controller, we derive an optimal controller that will penalize its high-gain portion. A numerical example illustrates the theoretical developments.

## Dynamics and Kinematics

We consider a rigid body with an axis of symmetry and two control torques about axes spanning the two-dimensional plane perpendicular to this axis. Without loss of generality we take a body-fixed reference frame $\boldsymbol{b} = (\hat{b}_1, \hat{b}_2, \hat{b}_3)$ with the unit vector $\hat{b}_3$ along the

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symmetry axis and the acting torques along the $\hat{b}_1$ and $\hat{b}_2$ axes. The Euler equations with respect to this frame then take the form

$$\dot{\omega}_1 = \frac{I_2 - I_3}{I_1}\omega_2\omega_3 + M_1 \tag{1a}$$

$$\dot{\omega}_2 = \frac{I_3 - I_1}{I_2}\omega_3\omega_1 + M_2 \tag{1b}$$

$$\dot{\omega}_3 = \frac{I_1 - I_2}{I_3}\omega_1\omega_2 \tag{1c}$$

For $I_1 = I_2$ and letting the initial condition $\omega_3(0) = \omega_{30}$, we can rewrite the preceding equations as

$$\dot{\omega}_1 = a\omega_{30}\omega_2 + u_1 \tag{2a}$$

$$\dot{\omega}_2 = -a\omega_{30}\omega_1 + u_2 \tag{2b}$$

where $a = (I_2 - I_3)/I_1$ and $u_i = M_i/I_i$, $(i = 1, 2)$.

If $\hat{\boldsymbol{n}} = (\hat{n}_1, \hat{n}_2, \hat{n}_3)$ denotes the inertial reference frame, then, as was shown in Ref. 20, the position of the $\hat{n}_3$ inertial axis in the $\hat{\boldsymbol{b}}$ frame can be uniquely described by two variables $w_1$ and $w_2$ that obey the differential equations

$$\dot{w}_1 = \omega_3 w_2 + \omega_2 w_1 w_2 + (\omega_1/2)\left(1 + w_1^2 - w_2^2\right) \tag{3a}$$

$$\dot{w}_2 = -\omega_3 w_1 + \omega_1 w_1 w_2 + (\omega_2/2)\left(1 + w_2^2 - w_1^2\right) \tag{3b}$$

This kinematic description is especially suitable for attitude description and control of axisymmetric bodies, where typically only the location of the symmetry axis is of interest. Thus $w_1$ and $w_2$ can be used to keep track of the deviation of the symmetry axis from the $\hat{n}_3$ inertial axis.

Equations (2) and (3) can be written in a vector form as

$$\dot{\omega} = aS(\omega_{30})\omega + u \tag{4a}$$

$$\dot{w} = S(\omega_{30})w + F(w)\omega \tag{4b}$$

where $\omega = [\omega_1 \;\; \omega_2]^T$, $w = [w_1 \;\; w_2]^T$, where $F : \mathbb{R}^2 \rightarrow \mathbb{R}^{2\times 2}$ is the symmetric, matrix-valued function defined by

$$F(w) = \tfrac{1}{2}[(1 - w^T w)I + 2ww^T] \tag{5}$$

and where $S(\omega_{30})$ is the $2 \times 2$ skew-symmetric matrix

$$S(\omega_{30}) = \begin{bmatrix} 0 & \omega_{30} \\ -\omega_{30} & 0 \end{bmatrix} \tag{6}$$

Given Eqs. (4), the main objective of this paper is to derive feedback control laws $u = u(\omega, w)$ that will drive $w$ and $\omega$ to zero in some optimal fashion. According to the preceding discussion, this amounts to reorienting the symmetry axis to a desired position optimally (assumed here to be the inertial axis $\hat{n}_3$).

## Equation Structure and Passivity

Equations (4) have the nice structure of a system in cascade form (see Fig. 1). That is, $w$ does not enter into the dynamics in Eq. (4a) and $u$ does not affect the kinematics in Eq. (4b). In fact, the kinematics can only be manipulated through appropriate choice of the angular velocity profile. This motivates the decomposition of the complete system into a dynamics and a kinematics subsystem. Another important property of the system in Eqs. (4) is that it represents a cascade interconnection of two passive systems. This allows for linear, globally asymptotically stabilizing control laws.

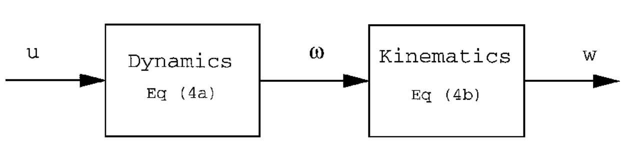


**Fig. 1 Cascade connection of dynamics and kinematics.**

Recall that a system with input $u \in \mathbb{R}^m$ and output $y \in \mathbb{R}^m$ is *passive* (with storage function $V$) if there exists a positive definite function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that[21,22]

$$\int_0^T y^T(t)u(t)\,\mathrm{d}t \geq V[x(T)] - V[x(0)] \tag{7}$$

where $x \in \mathbb{R}^n$ is the state of the system. If Eq. (7) is satisfied with equality, the system is *lossless*.[22]

A system is *strictly passive* (with storage function $V$ and dissipation rate $\psi$) if there exist positive definite functions $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ and $\psi : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that[21]

$$\int_0^T y^T(t)u(t)\,\mathrm{d}t \geq V[x(T)] - V[x(0)] + \int_0^T \psi[x(t)]\,\mathrm{d}t \tag{8}$$

In the following, $\|\cdot\|$ denotes the 2-norm, that is, $x^T x = \|x\|^2$ for any $x \in \mathbb{R}^n$.

*Proposition 1.* a) Consider the system in Eq. (4a) with input $u$ and output $\omega$. This system is passive with storage function

$$V_1(\omega) = \tfrac{1}{2}\|\omega\|^2 \tag{9}$$

b) Consider the system in Eq. (4b) with input $\omega$ and output $w$. This system is passive with storage function

$$V_2(w) = \ln(1 + \|w\|^2) \tag{10}$$

*Proof.* a) To show that the dynamics subsystem in Eq. (4a) is passive, notice that the derivative of $V_1$ in Eq. (9) along the trajectories of Eq. (4a) is

$$\frac{\mathrm{d}V_1}{\mathrm{d}t} = \omega^T u \tag{11}$$

Integrating both sides of the preceding equation from 0 to $T$, we arrive at Eq. (7).

b) To show that the kinematics subsystem in Eq. (4b) is passive, notice that the derivative of $V_2$ in Eq. (10) along the trajectories of Eq. (4b) is

$$\frac{\mathrm{d}V_2}{\mathrm{d}t} = w^T \omega \tag{12}$$

Integrating both sides, we arrive at Eq. (7). □

This proposition shows that the system in Eqs. (4) is a cascade interconnection of two passive systems. Passivity is invariant under feedback interconnection, but cascade interconnection of two passive systems is not necessarily passive. Nevertheless, it will be shown in this section that the cascade interconnection of two passive systems can always be globally asymptotically stabilized by linear feedback of the subsystem outputs. We will state and prove this result for the system interconnection (4a) and (4b). This result can easily be extended, however, to the case of a cascade interconnection of any two (nonlinear) passive systems.

*Lemma 1.* The control law

$$u = -k_1\omega + v \tag{13}$$

with $k_1 > 0$ renders the subsystem (4a) strictly passive from $v$ to $\omega$ with storage function $V_1$ and dissipation rate $\psi(\omega) = k_1\|\omega\|^2$.

*Proof.* Letting $V_1$ as in Eq. (9) and using Eqs. (11) and (13), we get that

$$\frac{\mathrm{d}V_1}{\mathrm{d}t} = -k_1\|\omega\|^2 + \omega^T v \tag{14}$$

Integrating both sides of this equation, one obtains

$$\int_0^T \omega^T v\,\mathrm{d}t = V_1[\omega(T)] - V_1[\omega(0)] + k_1\int_0^T \|\omega\|^2\,\mathrm{d}t \tag{15}$$

which, according to Eq. (8), implies that the system from $v$ to $\omega$ is strictly passive. □

This lemma shows that we have a cascade interconnection of a strictly passive system (from $v$ to $\omega$) with a passive system (from $\omega$

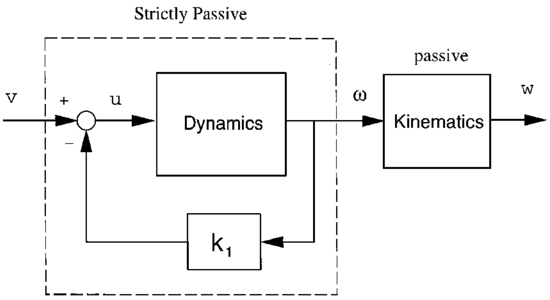


**Fig. 2 Passive interconnection with control $u = -k_1\omega + \nu$.**

to $w$); see Fig. 2. Choosing a negative feedback from $w$ to $\nu$ (e.g., $\nu = -k_2 w$), the resulting closed-loop system is then a feedback interconnection of a passive with a strictly passive system, and global asymptotic stability can be easily shown under an observability assumption, which in our case is satisfied. The following theorem formalizes this observation.

*Theorem 1.* Consider the cascade interconnection (4a) and (4b). The linear control

$$u = -k_1\omega - k_2 w \tag{16}$$

where $k_1,\ k_2 > 0$ globally asymptotically stabilizes this system.

*Proof.* Choosing the negative feedback $\nu = -k_2 w$, one obtains a feedback interconnection of a strictly passive system with a passive system. Therefore, by the passivity theorem,[21] the closed-loop system is globally asymptotically stable. To see this, let the positive definite, radially unbounded function

$$V(\omega,\ w) = V_1(\omega) + k_2 V_2(w) = \tfrac{1}{2}\|\omega\|^2 + k_2 \ln(1+\|w\|^2) \tag{17}$$

Taking the derivative of $V$ along the trajectories of Eqs. (4–16), one obtains

$$\dot V = \omega^T\dot\omega + \frac{2k_2}{1+\|w\|^2}w^T\dot w = -k_1\|\omega\|^2 - k_2\omega^T w$$
$$+\frac{k_2 w^T}{1+\|w\|^2}[S(\omega_{30})w + F(w)\omega] = -k_1\|\omega\|^2 \le 0 \tag{18}$$

and the system is stable. Asymptotic stability follows using a standard LaSalle-type argument.[23] □

## Relation Between Optimality and Passivity

In the next section we will address the optimal regulation problem for the system in Eqs. (4) subject to a quadratic cost. We will show that the two optimal control problems in terms of the dynamics and the kinematics subsystems have a closed-form solution. In this section we show that this remarkable result is not accidental but stems from the passivity properties of the corresponding subsystems. In particular, we will show that if a nonlinear system is lossless (passive), then there exists a control law that is optimal (gives an upper bound) with respect to a quadratic cost in the state and the control input. Moreover, this optimal (suboptimal) control law is linear.

Consider the nonlinear system

$$\dot x = f(x,u), \qquad x\in\mathbb{R}^n, \qquad u\in\mathbb{R}^n \tag{19}$$

where $f:\mathbb{R}^{n\times n}\to\mathbb{R}^n$ is a smooth vector field such that $f(0,0) = 0$. Let us assume that the system in Eq. (19) is passive from $u$ to the state $x$ with storage function $V$. The following theorem states the main result between optimality and passivity used in this paper.

*Theorem 2.* Let the system in Eq. (19) and the quadratic cost

$$\mathcal{J} = \tfrac{1}{2}\textstyle\int_0^\infty \left\{r_1\|x\|^2 + r_2\|u\|^2\right\}\mathrm{d}t \tag{20}$$

where $r_1$ and $r_2$ are positive scalars. Then the linear control law

$$u^*(x) = -\sqrt{r_1/r_2}\,x \tag{21}$$

provides an upper bound for the cost in Eq. (20).

*Proof.* Consider any stabilizing control law $u$ and let $v = u - u^*$. The cost in Eq. (20) can then be written as

$$\mathcal{J} = \frac{1}{2}\int_0^\infty \left\{r_1\|x\|^2 + r_2\|v+u^*\|^2\right\}\mathrm{d}t$$
$$= \frac{1}{2}\int_0^\infty \left\{r_1\|x\|^2 + r_2\|v\|^2 + r_2\|u^*\|^2 + 2r_2 v^T u^*\right\}\mathrm{d}t \tag{22}$$

Using the expression for $u^*(x)$ from Eq. (21), one obtains

$$\mathcal{J} = \frac{1}{2}\int_0^\infty \left\{2r_1\|x\|^2 + r_2\|v\|^2 - 2\sqrt{r_1r_2}\,v^T x\right\}\mathrm{d}t$$
$$= \frac{1}{2}\int_0^\infty \left\{2r_1\|x\|^2 + r_2\|v\|^2 - 2\sqrt{r_1r_2}(u-u^*)^T x\right\}\mathrm{d}t$$
$$= \frac{1}{2}\int_0^\infty r_2\|v\|^2\,\mathrm{d}t - \sqrt{r_1r_2}\int_0^\infty u^T x\,\mathrm{d}t \tag{23}$$

The control law $u(x)$ is stabilizing; thus $\lim_{t\to\infty} x(t) = 0$. Using Eq. (7) now and the fact that $\lim_{t\to\infty} V[x(t)] = 0$, one finally obtains

$$\mathcal{J} \le \frac{1}{2}\int_0^\infty r_2\|v\|^2\,\mathrm{d}t + \sqrt{r_1r_2}\,V[x(0)] \tag{24}$$

For $v = 0$, we have

$$\mathcal{J} = \frac{1}{2}\int_0^\infty \left\{r_1\|x\|^2 + r_2\|u\|^2\right\}\mathrm{d}t \le \sqrt{r_1r_2}\,V[x(0)] \tag{25}$$

and the control law $u^*(x)$ provides an upper bound for $\mathcal{J}$, as claimed. □

If the system is, in fact, lossless, we get the following optimality result, which is given without proof.

*Corollary 1.* Assume that the system in Eqs. (19) is lossless. Then the linear control law in Eq. (21) is optimal with respect to the cost in Eq. (20). Moreover, the minimum value of the cost is $\min_u \mathcal{J}[u;\ x(0)] = \sqrt{(r_1r_2)}\,V[x(0)]$.

We note in passing that Theorem 2 is rather restrictive the way it is stated here because it requires $y = x$. However, the same result will also hold for the more general case when $y \ne x$. The optimal controller will then be linear in $y$, whereas asymptotic stability will require an extra observability condition.

## Optimal Regulation

### Kinematics Subsystem

Given the kinematics system in Eq. (4b), where $\omega$ is treated as a control-like variable, we introduce the following performance index:

$$\mathcal{J}_1(w,\omega) = \frac{1}{2}\int_0^\infty \left\{r_1\|w(t)\|^2 + r_2\|\omega(t)\|^2\right\}\mathrm{d}t \tag{26}$$

where $r_1$ and $r_2$ are positive constants. Notice that this functional is a true performance index in the sense that it penalizes the state ($w$) and the control input ($\omega$).

According to the Hamilton–Jacobi theory, the optimal feedback control $\omega^*$ for the preceding problem is given by

$$0 = \min_\omega\left\{\frac{r_1}{2}\|w\|^2 + \frac{r_2}{2}\|\omega\|^2 + \frac{\partial V}{\partial w}[S(\omega_{30})w + F(w)\omega]\right\} \tag{27}$$

where $\partial V/\partial w$ denotes the gradient of $V$ (row vector). Therefore, the HJE associated with the optimal control problem in Eqs. (4b) and (26) is given by

$$\frac{r_1}{2}\|w\|^2 - \frac{1}{2r_2}\left\|F(w)\frac{\partial^T V}{\partial w}\right\|^2 + \frac{\partial V}{\partial w}S(\omega_{30})w = 0 \tag{28}$$

and the optimal control is given by

$$\omega^*(w) = -\frac{1}{r_2}F(w)\frac{\partial^T V}{\partial w} \tag{29}$$

We claim that the positive definite function $V : \mathbb{R}^2 \to \mathbb{R}_+$ defined by

$$V(w) = \sqrt{r_1 r_2}\, \ln(1 + \|w\|^2) \tag{30}$$

solves the Eq. (28). Indeed, noticing that

$$\frac{\partial V}{\partial w} = \frac{2\sqrt{r_1 r_2}}{1 + \|w\|^2}\, w^T \tag{31}$$

and that

$$F(w)\frac{\partial^T V}{\partial w} = \sqrt{r_1 r_2}\, w \tag{32}$$

substituting in Eq. (28), and using the fact that $w^T S(\omega_{30}) w = 0$, we obtain the desired result. The optimal control is given by Eq. (29) and takes the very simple form

$$\omega^*(w) = -\sqrt{r_1/r_2}\, w \tag{33}$$

Note that the optimal control in Eq. (33) is linear and it is unique. Moreover, using $V$ from Eq. (30) as a Lyapunov function for the closed-loop system, it is not difficult to show that the optimal control is exponentially stabilizing. The minimum value of the cost in Eq. (26) is given by

$$\mathcal{J}_1^*[w(0)] = \sqrt{r_1 r_2}\, \ln(1 + \|w(0)\|^2) = V[w(0)] \tag{34}$$

**Dynamics Subsystem**

So far, we have only considered the kinematics subsystem or the attitude equations, i.e., Eq. (4b), with $\omega$ acting as a control variable. The optimal regulation problem for the Eq. (4a) has been addressed and solved elsewhere.[1] We only state the result for completeness, without proof.

To this end, consider the system in Eq. (4a), where $u$ is the control input, and let the quadratic performance index

$$\mathcal{J}_2(\omega, u) = \frac{1}{2}\int_0^\infty \left\{q_1\|\omega(t)\|^2 + q_2\|u(t)\|^2\right\} \mathrm{d}t \tag{35}$$

where $q_1$ and $q_2$ are positive constants. Then the control law

$$u^*(\omega) = -\sqrt{q_1/q_2}\, \omega \tag{36}$$

renders the closed-loop system globally exponentially stable at the origin and minimizes the cost in Eq. (35). Moreover, the minimum value of this cost is

$$\mathcal{J}_2^*[\omega(0)] = \tfrac{1}{2}\sqrt{q_1 q_2}\, \|\omega(0)\| \tag{37}$$

**Complete System**

We have considered the kinematics and the dynamics subsystems of the attitude equations separately. The natural question is, of course, "what conclusions can be drawn about the complete system interconnection?" Previous attempts include approximate solutions using truncated Taylor series expansions of the HJE[13] or exact solutions of a feedback linearized version of the problem.[17] The feedback linearization technique is especially appealing but has the drawback that the optimization is performed in the transformed variables (which may not be directly amenable to a physical interpretation) and that the penalty on the control does not include the feedback linearizing portion.

Our approach is based on the observation that we already have an exact solution of the optimal regulation problem for the kinematics. We wish to use this knowledge from the kinematics problem instead of formulating an entirely new problem for the complete system. This approach limits our freedom in choosing the performance index but allows the analytic derivation of optimal feedback controllers in closed form.

If the dynamics subsystem is sufficiently fast, then the previous optimality results suffice. In this case, the optimal angular velocity profile can be implemented through the dynamics without significant degradation in performance. Actually, one can always recover the cost in Eq. (34) asymptotically, using the control input

$$u_{as} = -aS(\omega_{30})\omega - kF(w)\omega - kS(\omega_{30})w - \lambda(\omega + kw) \tag{38}$$

where $k = \sqrt{r_1/r_2}$. That is, by choosing $\lambda$ large enough, the cost

$$\int_0^\infty \left\{r_1\|w\|^2 + r_2\|\omega\|^2\right\} \mathrm{d}t \to \sqrt{r_1 r_2}\, \ln(1 + \|w(0)\|^2) \tag{39}$$

and it can be made arbitrarily close to $\mathcal{J}_1^*[w(0)]$.

This result can be shown by introducing the new variable

$$\zeta = \omega + kw \tag{40}$$

and rewriting the system in Eq. (4) with the control in Eq. (38) in the form

$$\dot{\zeta} = -\lambda\zeta \tag{41a}$$

$$\dot{w} = S(\omega_{30})w - kF(w)w + F(w)\zeta \tag{41b}$$

Notice from Eq. (40) that since $\zeta \to 0$, then $\omega \to \omega^*$. We can explicitly calculate the value of the cost $\mathcal{J}_1(w, \omega)$ along the trajectories of Eqs. (41) using the positive definite function

$$\begin{aligned} V(w, \omega) &= 2\sqrt{r_1 r_2}\, \ln(1 + \|w\|^2) + (r_2/2\lambda)\|\omega + kw\|^2 \\ &= 2\sqrt{r_1 r_2}\, \ln(1 + \|w\|^2) + (r_2/2\lambda)\|\zeta\|^2 \end{aligned} \tag{42}$$

Then

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{4\sqrt{r_1 r_2}}{1 + \|w\|^2} w^T[-kF(w)w + F(w)\zeta] - r_2\|\zeta\|^2 \\ &= -2\sqrt{r_1 r_2}\, k\|w\|^2 + 2\sqrt{r_1 r_2} w^T\zeta - r_2\|\zeta\|^2 \\ &= -r_1\|w\|^2 - r_2\|\zeta - kw\|^2 \\ &= -r_1\|w\|^2 - r_2\|\omega\|^2 \le 0 \end{aligned} \tag{43}$$

Since $\dot{V}$ is negative definite, the control law in Eq. (38) is asymptotically stabilizing. Thus $\lim_{T\to\infty} V(T) = 0$. Integrating both sides and taking limits as $T \to \infty$, one obtains

$$V[w(0), \omega(0)] = \int_0^\infty \left\{r_1\|w\|^2 + r_2\|\omega\|^2\right\} \mathrm{d}t \tag{44}$$

Since

$$\begin{aligned} V[w(0), \omega(0)] &= 2\sqrt{r_1 r_2}\, \ln(1 + \|w(0)\|^2) \\ &\quad + (r_2/2\lambda)\|\omega(0) + kw(0)\|^2 \end{aligned} \tag{45}$$

then Eq. (39) follows by letting $\lambda \to \infty$. A simple singular perturbation analysis shows that the effect of large $\lambda$ is that of making the dynamics in Eq. (4a) sufficiently fast.

The optimal cost in Eq. (26) provides a lower bound on the achievable performance when the actual control input is the body-fixed torque $u$. The disadvantage of the control law in Eq. (38) is that it may require high gain. This may not be acceptable if there are bounds on the available control effort. A more realistic performance index should incorporate a penalty on the control effort $u$ as well. Unfortunately, the optimization problem for a performance index that is quadratic in the state and the control effort is rather formidable. Motivated by the control law in Eq. (38), we use an alternative approach. We investigate the optimality properties of the control law in Eq. (38), and, in particular, we modify this control law such that its high-gain portion is penalized.

The procedure in this section is similar in spirit to the results of Ref. 24, where the authors examine the optimality properties of a class of feedback control laws for relative degree one minimum phase systems, and the results in Ref. 18, where the optimal regulation problem for a general, i.e., nonsymmetric, body is addressed.

Close examination of the control law in Eq. (38) shows that the only possible high-gain portion of this control law is the last term. We therefore consider a modified control law of the form

$$u = -aS(\omega_{30})\omega - kF(w)\omega - kS(\omega_{30})w + v \tag{46}$$

Recalling now the desirable properties of the relationship $\omega = -kw$ for the kinematic subsystem, we again introduce the variable $\zeta = \omega + kw$ and develop control laws that will make $\zeta \to 0$. The performance index should therefore include a penalty on $\zeta$ as well as a penalty on the control effort $v$.

Using Eqs. (46) and (40), the system in Eqs. (4) is written in the form

$$\dot{\zeta} = v \tag{47a}$$

$$\dot{w} = S(\omega_{30})w + F(w)(\zeta - kw) \tag{47b}$$

*Theorem 3.* Consider the system in Eqs. (47) and the control law

$$v^*(w, \zeta) = -(w/\lambda) - \lambda\zeta \tag{48}$$

Then this control law makes the system in Eq. (47) exponentially stable and minimizes the cost

$$\mathcal{J}_3(w, \zeta, v) = \frac{1}{2}\int_0^\infty \left\{ \left\| v + \frac{w}{\lambda} \right\|^2 + 2k\|w\|^2 + \lambda^2\|\zeta\|^2 \right\} \mathrm{d}t \tag{49}$$

Moreover, the minimum value of the cost is

$$\mathcal{J}_3^*[w(0), \zeta(0)] = \ln(1 + \|w(0)\|^2) + (\lambda/2)\|\zeta(0)\|^2 \tag{50}$$

*Proof.* First, notice that the HJE associated to the preceding optimal control problem is given by

$$\frac{1}{2}\left\|\frac{\partial V}{\partial \zeta}\right\|^2 + k\|w\|^2 + \frac{\lambda^2}{2}\|\zeta\|^2 - \frac{\partial V}{\partial \zeta}\left(\frac{w}{\lambda} + \lambda\zeta\right) - \frac{\partial V}{\partial w}S(\omega_{30})w + \frac{\partial V}{\partial w}F(w)(\zeta - kw) = 0 \tag{51}$$

and the optimal control is given by

$$v^*(w, \zeta) = -\frac{w}{\lambda} - \frac{\partial^T V}{\partial \zeta} \tag{52}$$

Then notice that the positive definite function $V_3 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_+$ defined by

$$V_3(w, \zeta) = \ln(1 + \|w\|^2) + (\lambda/2)\|\zeta\|^2 \tag{53}$$

satisfies the HJE in Eq. (51). The exponential stabilizability of the control in Eq. (48) is easily verified by using Eq. (53) as a Lyapunov function for the closed-loop system. The minimum value of the cost is given by $\mathcal{J}_3^*[w(0), \zeta(0)] = V_3[w(0), \zeta(0)]$. □

From Eqs. (46) and (52), we have that the optimal control is

$$u^*(\omega, w) = -aS(\omega_{30})\omega - kF(w)\omega - kS(\omega_{30})w - \lambda(\omega + kw) - (w/\lambda) \tag{54}$$

Moreover,

$$u^* = u_{as} - (w/\lambda) \tag{55}$$

Comparison of Eqs. (48) and (49) shows that the control law

$$\tilde{v}^*(\zeta) = -\lambda\zeta \tag{56}$$

minimizes the cost

$$\tilde{\mathcal{J}}_3(w, \zeta, \tilde{v}) = \frac{1}{2}\int_0^\infty \{\|\tilde{v}\|^2 + 2k\|w\|^2 + \lambda^2\|\zeta\|^2\} \, \mathrm{d}t \tag{57}$$

subject to the dynamical constraints

$$\dot{\zeta} = -(w/\lambda) + \tilde{v} \tag{58a}$$

$$\dot{w} = S(\omega_{30})w + F(w)(\zeta - kw) \tag{58b}$$

That is, the first term in Eq. (57) includes a true penalty on the high-gain portion of the controller. Moreover, notice that as $\lambda \to \infty$, then $v^* \to -\lambda\zeta$ and $u^* \to u_{as}$, and we recover the results of the control law in Eq. (38). In essence, the control law $u^*$ allows one to decrease $\lambda$ without significant degradation in the stability and performance. As it is evident from Eq. (57) the parameter $\lambda$ can be chosen to compromise between good performance (in the sense of small $\zeta$) and acceptable control gain.

## Numerical Example

We illustrate the theoretical results by means of numerical simulations. We consider an optimal regulation maneuver of an axisymmetric rigid body from initial orientation $w_1(0) = w_2(0) = 10$. These values correspond to a rigid body that is, initially, almost "upside down." The body is assumed to be initially at rest. Therefore, $\omega_1(0) = \omega_2(0) = \omega_3(0) = 0$. The inertia parameter in Eq. (2) is $a = 0.5$. The constants $r_1$ and $r_2$ in Eq. (33) were chosen to be equal to unity, which implies that also $k = 1$ in Eq. (40). The control law in Eq. (54) is implemented for different values of $\lambda$.

The results are shown in Figs. 3–5. Figures 3 and 4 show the response for the first component of the angular velocity and the orientation parameter $w$, respectively. The control effort for different values of $\lambda$ is shown in Fig. 5. Decreasing the value of $\lambda$ has the effect of increasing the oscillatory behavior of the system, but the rate of convergence seems to remain relatively constant.

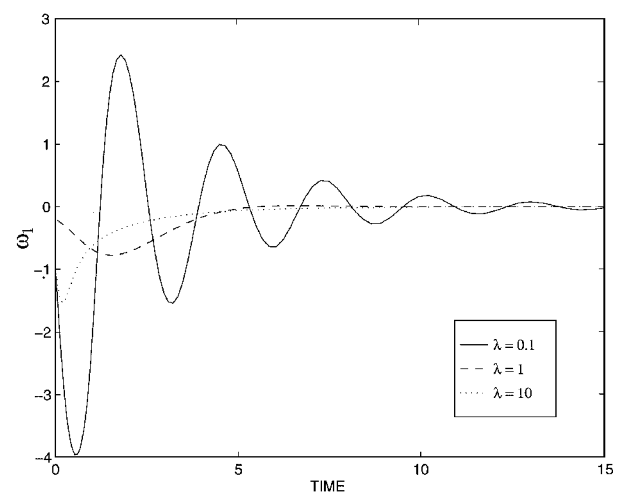


**Fig. 3 Angular velocity response using $u^*$.**

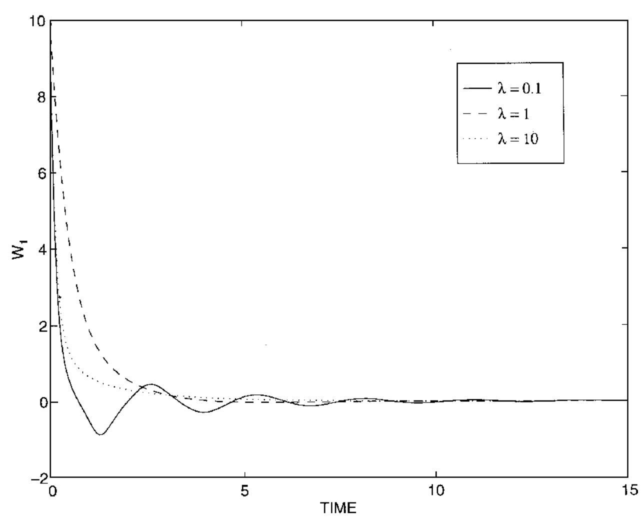


**Fig. 4 Orientation parameter response using $u^*$.**

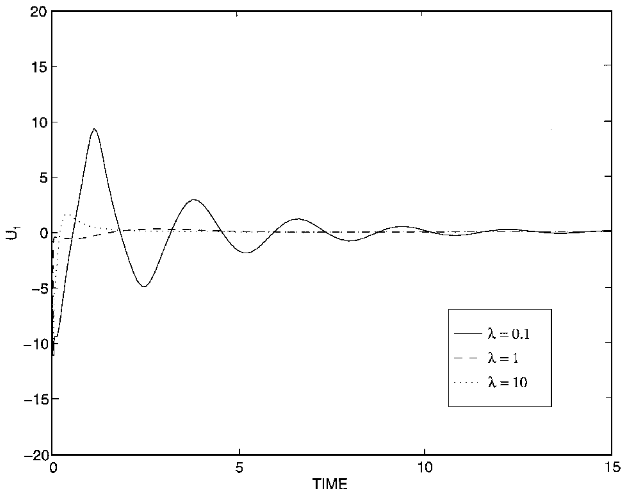


**Fig. 5 Control input response using $u^*$.**

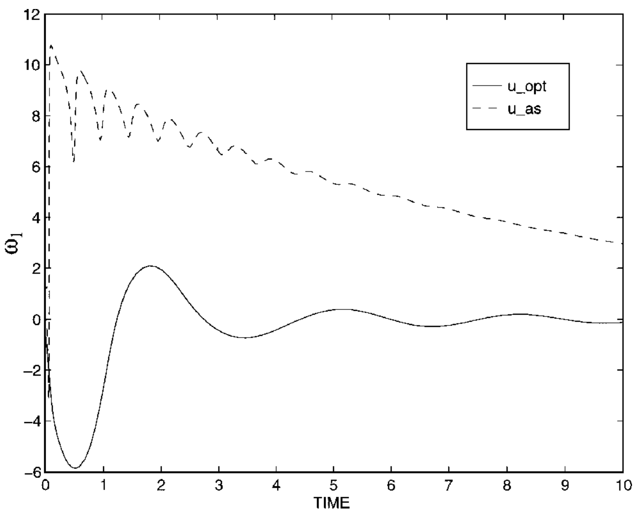


**Fig. 6 Comparison of angular velocity for $u^*$ and $u_{as}$.**

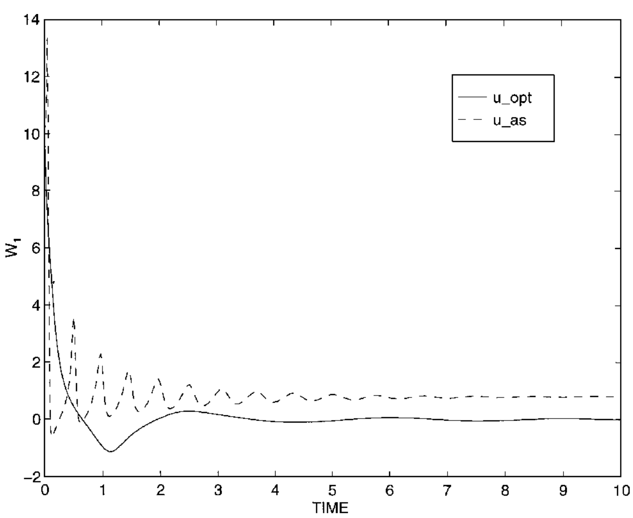


**Fig. 7 Comparison of orientation parameters for $u^*$ and $u_{as}$.**

Figures 6–8 compare the control laws $u^*$ and $u_{as}$ for a small value of the gain ($\lambda = 0.1$). Recall that for large $\lambda$ these two control laws are essentially the same. The initial conditions for the orientation are as before, and the initial conditions for the angular velocity are given by $\omega_1(0) = \omega_2(0) = 0.25\, r/s$ and $\omega_3(0) = 0.1\, r/s$. These simulations were typical—at least for the range of initial conditions and gains checked—of the relative response of the two controllers. They seem to verify that the optimal control law $u^*$ performs better than the asymptotic control law $u_{as}$. Note in particular in Figs. 6 and 7 the slow convergence rates of the states for the controller $u_{as}$.

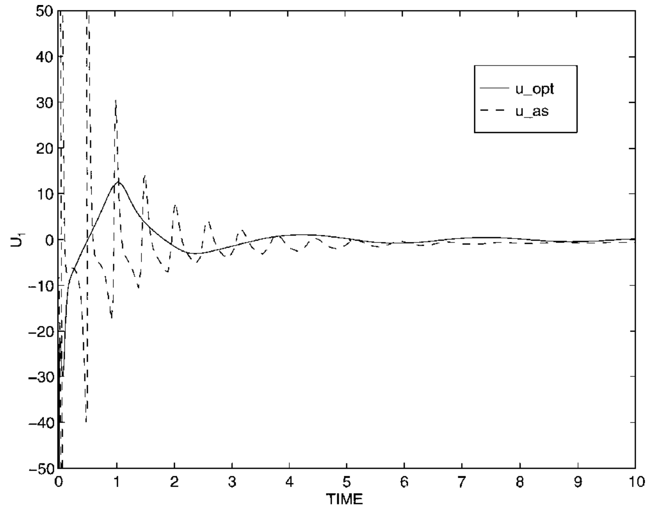


**Fig. 8 Comparison of control input for $u^*$ and $u_{as}$.**

## Conclusions

We have presented some new results for the optimal regulation of the symmetry axis of a spinning rigid body. Only two control torques are necessary if regulation of the relative rotation about the symmetry axis is not required. By using the natural decomposition of the system into its kinematics and dynamics subsystems and the inherent passivity properties of these two subsystems, we derived an optimal controller in a two-step process. The optimal control for the kinematics is very simple (linear) and minimizes a quadratic cost in terms of the angular velocity and the kinematic parameters. The derivation of this optimal controller is intimately connected to the passivity of the kinematics. Direct implementation of this control through the dynamics may require high gain, however. We modified this direct approach to obtain an optimal controller that tries to mimic the optimal controller for the kinematics by penalizing its high-gain portion. The gain parameter can be used to compromise between speed of regulation and acceptable control effort.

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